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*Engineering a passive Non-linear automobile suspension*

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**Abstract**

Most passive suspensions are designed with linear spring and dampers. This paper presents a novel design for an automobile suspension system for a highly fuel efficient, ultra-low weight passenger car weighing approximately 600kg. In this paper the effects of non-linear spring and damping were studied.

According to <sup>[1]</sup> the ride comfort perceived by passengers has been shown to depend on the r.m.s acceleration of the chassis and also on rate of change of acceleration. A good suspension system would stiffen up for larger rates and soften for smaller rates of acceleration input from the road when the car moves over bumps. This is related to certain desirable nonlinearity both in spring as well as in damper <sup>[2]</sup>. Active suspensions can provide superior performance because they bring about this nonlinearity that is tuned actively. But, passive systems could also be tuned nonlinearly, although it would not be possible to change the algorithm continuously as per road input as in active systems. Still, some significant improvements could be brought about if the response of the system could depend on the rate of acceleration felt by the chassis. This theoretical possibility has been realized in this paper with a practically possible passive suspension design with non-linear dampers. The performance of this design is compared with that of conventional suspension system with similar mass and stiffness elements on the basis of the key design metrics – ride comfort, steering control, road holding – as modelled in <sup>[3]</sup>.

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### 1.0 Introduction

Most passive suspensions are designed with linear spring and dampers. This paper presents a novel design for an automobile suspension system for a highly fuel efficient, ultra-low weight passenger car weighing approximately 600kg. In this paper the effects of passive non-linear damping was studied.

According to <sup>[1]</sup> the ride comfort perceived by passengers has been shown to depend on the r.m.s acceleration of the chassis and also on rate of change of acceleration. A good suspension system would stiffen up for larger rates and soften for smaller rates of acceleration input from the road when the car moves over bumps. This is related to certain desirable nonlinearity both in spring as well as in damper <sup>[2]</sup>. Active suspensions can provide superior performance because they bring about this nonlinearity that is tuned actively. But, passive systems could also be tuned nonlinearly, although it would not be possible to change the algorithm continuously as per road input as in active systems. Still, some significant improvements could be brought about if the response of the system could depend on the rate of acceleration felt by the chassis. This theoretical possibility has been realized in this paper with a practically possible passive suspension design with non-linear dampers. The performance of this design is compared with that of conventional suspension system with similar mass and stiffness elements on the basis of the key design metrics – ride comfort, steering control, road holding – as modelled in <sup>[3]</sup>.

Firstly, a schematic design is presented to account for the desired nonlinearity. It is known from <sup>[1]</sup>, <sup>[3]</sup> that damping being higher in the forward stroke than the return stroke improves the performance of the car both in terms of road holding and particularly, the ride quality. Considering this observation, various designs are evolved that progressively vary damping constant  $c$  as some function of displacement  $x$  but passively.

It will be seen that while spring itself kept linear, certain nonlinear damper design can itself bring about fair improvement in the ride quality. Ideal drive when a step input is encountered would be to just quickly reach the corresponding equilibrium position, and just as when it about to overshoot, the damping function so changes suddenly that oscillations are not seen. The passive damper design presented here does achieve this but at relatively larger strokes.

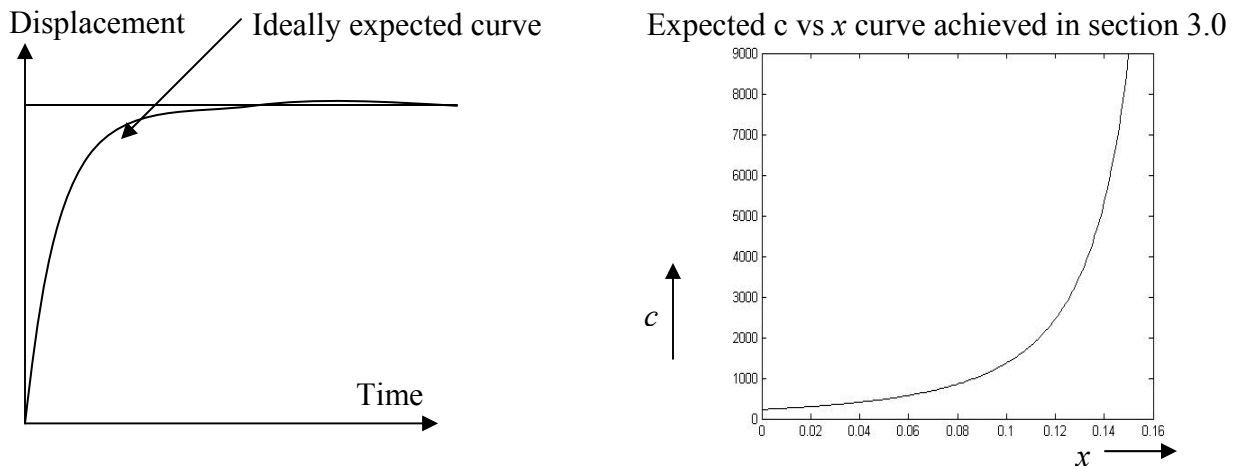


Figure 1 Ideally expected response and that achieved practically

## 2.0 Schematic design of conical dashpot with cylindrical piston

In this section, the schematic design that has damping constant  $c$  varying with the displacement  $x$  is presented and is analysed in the next section.

According to [1], the switchable damper type semiactive suspension system is of practical significance and has shown lot of promise in the recent past. In this, the forward stroke has a damping different than that of return stroke, and it is established [1] that forward stroke having higher  $c$  than return stroke improves the vehicle performance a lot. But, as can be seen, this is not continuously variable, it is only 2-way.

Therefore, considering that forward stroke is progressively damped higher and higher as the stroke proceeds and in the return stroke, damping progressively decreases is the requirement to improve vehicle performance, the following design is established:

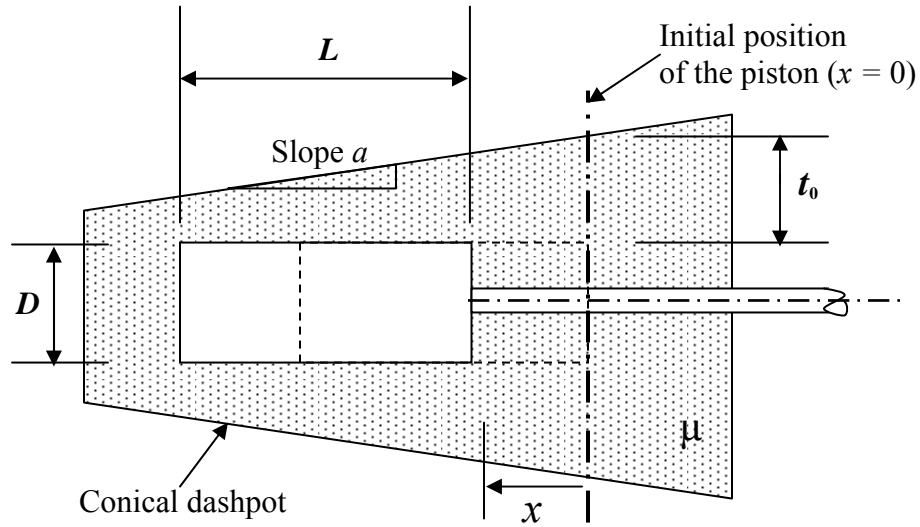


Figure 2 Conical dashpot with a cylindrical piston – schematic design

A cylindrical piston of diameter  $D$  is allowed to move inside a conical dashpot of slope  $a$ . At the starting position, that is, at  $x = 0$ , the clearance between the rear end of the piston and the dashpot is  $t_0$  as shown in figure.

The analytical expression for the variation of  $c$  with  $x$  is given by

$$c = \frac{3\pi\mu D^3}{4a} \left[ \frac{1}{2} \left( \frac{1}{(t_0 - a(L+x))^2} - \frac{1}{(t_0 - ax)^2} \right) + \frac{2}{D} \left( \frac{1}{t_0 - a(L+x)} - \frac{1}{t_0 - ax} \right) \right]$$

The derivation of this expression is presented in the Appendix section A.2.0 at the end.

NOTE: The derivation is done as per [5]

### 3.0 Quarter car simulation and discussion of results

The schematic damper design is tested for performance in the quarter car suspension system model depicted below in Figure 3.

For the purpose of comparison of performance, the linear model considered in [1] is used. That is,

- $k = 20 \text{ kN/m}$ ,  $k_t = 200 \text{ kN/m}$ ,  $c_t = 300 \text{ Ns/m}$ ,  $c = 1837 \text{ Ns/m}$  are used
- Although the design is intended for an ultra low weight car weighing around 600 kg, to avail comparison with available literature, mass of the car considered was 800 kg, hence, for the quarter car,  $m_s$  is taken as 200 kg. And, unsprung mass  $m_{us}$  is taken as 30 kg.
- Road input is simulated using response  $x_0(t)$  of a first order filter to white noise input  $\eta(t)$ :

$$\dot{x}_0 = -\alpha x_0 + \eta(t) \quad \dots (I)$$

That is, the solution to above equation (I) for  $x_0(t)$  is fed as road input to the quarter car model shown in Figure 3. Here,  $\alpha$  is the road roughness parameter which is 0.151/m for smooth road and 0.451/m for rough road.

- Most importantly, according to [1], the following are the evaluation metrics that were used to compare various models:

- Passenger comfort is measured by *rms acceleration of the sprung mass*
- Road holding capability is measured by *ratio of Dynamic tyre deflection to static tyre deflection*
- Space occupied is measured by *rms suspension deflection*
- Settling time shall be no more than 2 to 3 oscillations (approx. 1.5s) [4]

..... (II)

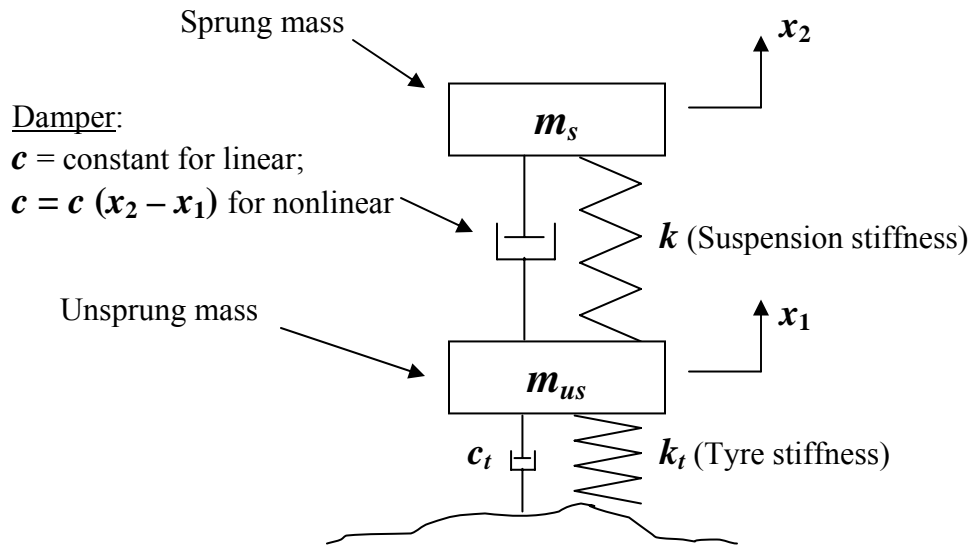
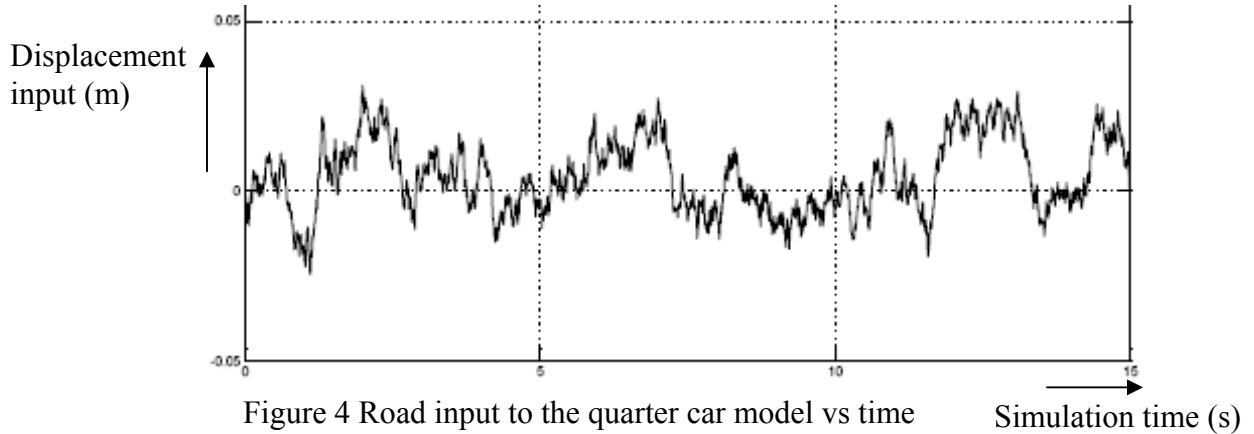


Figure 3 Quarter car schematic for simulation

### 3.1 Road profile

As explained above, road input is given as response to white noise of first order filter of  $\alpha = 0.151/\text{m}$ , Velocity  $v = 20\text{m/s}$ . This road profile is the common input to both the linear and nonlinear models. The maximum displacement input so given is  $0.0292\text{m}$ . Note that, like in linear systems, analysis can't be done in frequency domain since the system is nonlinear. Hence, time domain analysis becomes inevitable.

*All the models are tested for 15s of road profile input (i.e., for simulink time of 15s)*



Note :

1. Simulink toolbox in MATLAB 7.0 has been used for simulating various designs
2. Simulink Response Optimization has been used for tuning various parameters and optimising the system performance.

### 3.2 Response of linear suspension to the road profile

#### 3.2 (a) RMS sprung mass acceleration

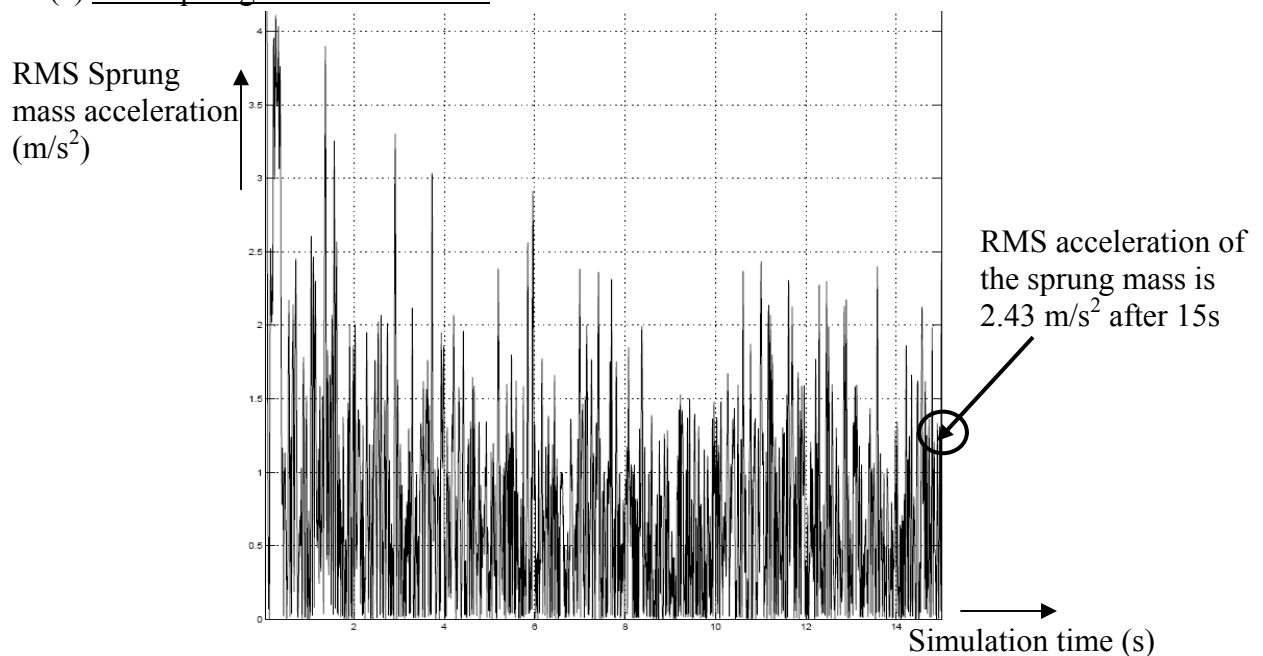


Figure 5 RMS sprung mass acceleration ( $\text{m/s}^2$ ) vs Simulation time (s)

### 3.2 (b) RMS tyre dynamic deflection

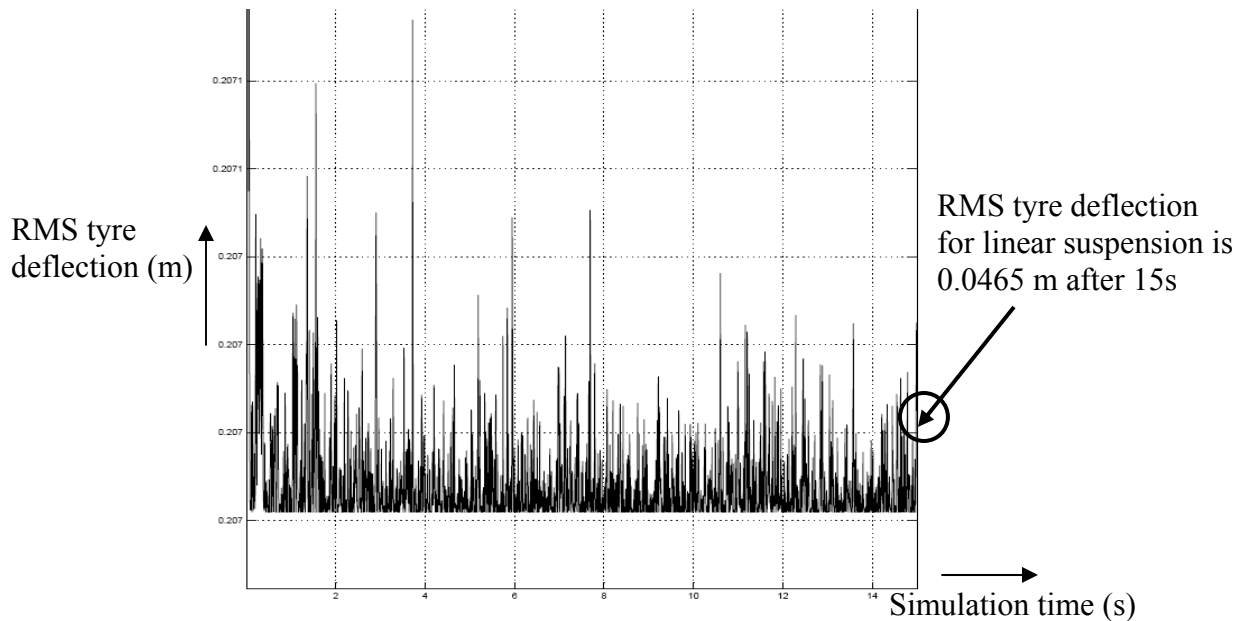


Figure 6 RMS tyre deflection for linear suspension vs simulation time

Note: Here, the y – axis values are  $\sim 0.207\text{m}$  – this is rms of the absolute position of unsprung mass. Hence, this must be subtracted from the equilibrium position  $0.2535\text{m}$  to get  $0.0465\text{m}$ . The static tyre deflection is the deflection that occurs when maximum value of the road input in road profile 4.1 is given as constant input. This was seen to be  $0.2827 - 0.2535 = 0.0292\text{m}$  Therefore, the road holding parameter = ratio of above two deflections =  $0.0465/0.0292 = 1.6$  for the linear suspension considered.

### 3.2 (c) RMS suspension deflection

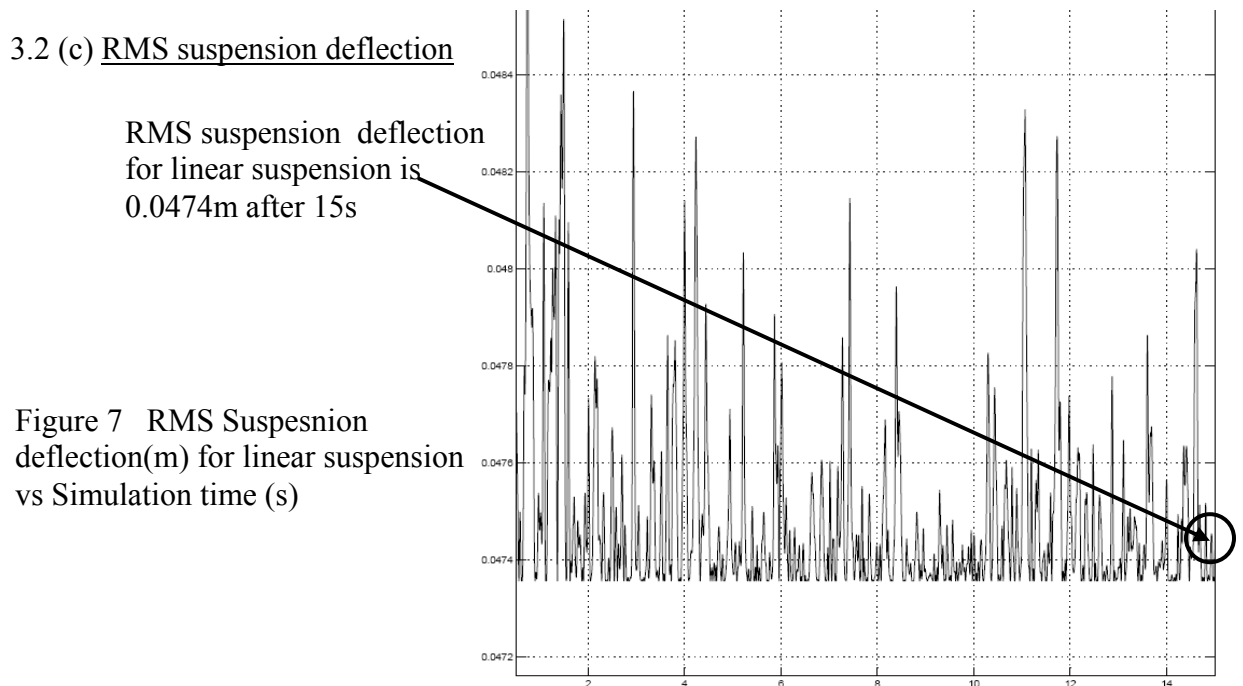


Figure 7 RMS Suspension deflection(m) for linear suspension vs Simulation time (s)

### 3.3 Evaluation of schematic design 2.0 – Conical dashpot with cylindrical piston damper

Making sure that metric  $D$  in (II) is met, that is, the oscillations under step input die down in 2 oscillations, the system is tuned to have Piston diameter  $D = 0.095\text{m}$ , Slope of the dashpot's conical surface  $a/2 = 0.0175$  (or  $a = 0.035$ ), Clearance at the start of stroke  $t_0 = 10$  mm, Length of piston immersed  $L = 0.1\text{m}$ , Oil viscosity  $\mu = 0.5$  Pa.s [Refer to Figure 2]

#### 3.3 (a) RMS sprung mass acceleration for the nonlinear damper

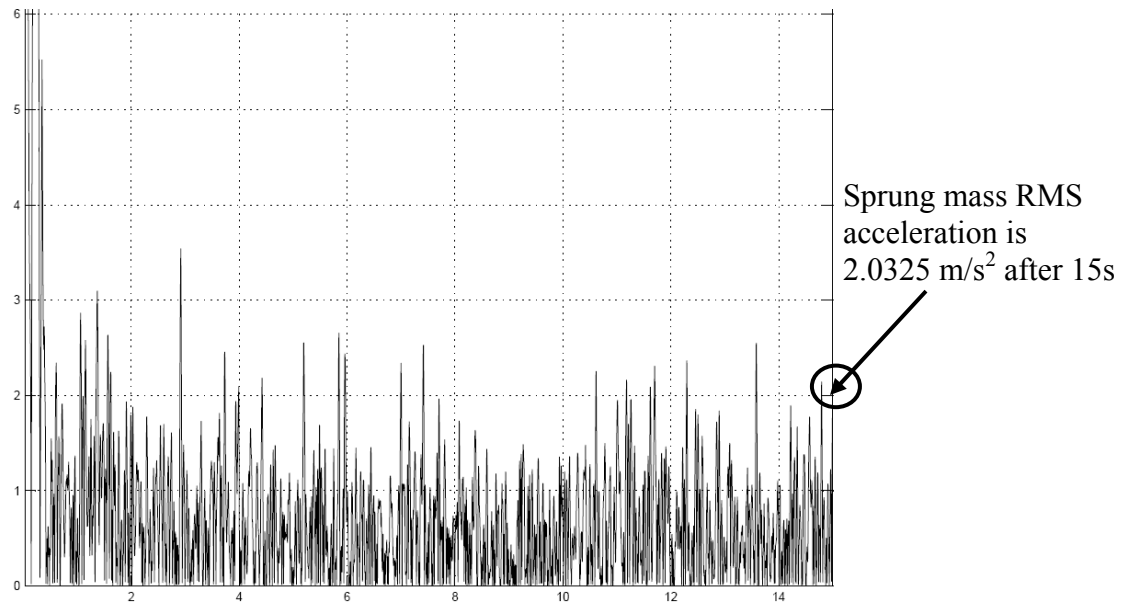


Figure 8 RMS sprung mass acceleration ( $\text{m/s}^2$ ) nonlinear damper vs Simulation time (s)  
– ends at  $2.0325 \text{ m/s}^2$  after 15s of simulation

After running the simulation for 15s, it was observed that the RMS sprung mass acceleration with this new damper is  $2.0325\text{m/s}^2$  which is almost 16.4% reduction from the linear case of  $2.43\text{m/s}^2$ .

For this design, the dynamic tyre deflection and the suspension deflection were observed to remain same as linear suspension but the rms acceleration could be dramatically reduced by 16.4% of the corresponding value of the linear case.

Hence, this design can be explored for further studies and detailed analysis

## 4.0 Conclusions

Hence, considering that the system must settle in utmost 2 oscillations<sup>[4]</sup>, with same RMS suspension deflection (hence similar space requirements), RMS tyre deflection (hence similar road holding); the new damper could be tuned to achieve 16.4% reduction in RMS acceleration of the sprung mass. Other constraints include – Dashpot diameter was not to exceed 10cm; There must be at least a clearance of 3 mm between the piston and the conical dashpot at the maximum stroke of the piston; Oil viscosity must be such that the oil must be available easily.

But, with a well designed objective function that strikes balance between road holding, ride comfort and space requirements, this can be further optimised to suit such requirements. As a subject of future work, suspension spring can also be made nonlinear and combined effects could then be studied.

## 5.0 References

- [1] Elmadany M M. and El-Tamimi A., *On a subclass of nonlinear passive and semi-active damping for vibration isolation*, Computers & Structures Vol. 36. No. 5. pp. 921-931, 1990
- [2] M.Tech. Theses, May 2006 , Mariappan S., *Modelling and dynamic simulation of landing gear for reusable launch vehicle*, Indian Institute of Technology, Madras.
- [3] Lane R Miller, *Tuning passive, semi-active and fully active suspension systems*, Proc. of 27<sup>th</sup> Conf. on Decision and Control, Dec. 1988, Austin, Texas
- [4] Colin Campbell, *Automobile suspensions*, Chapman and Hall Publishers, London
- [5] S.S.Rao, *Mechanical Vibrations*, Fourth edition, Pearson education



## Appendix

### A.1.0 Expression for damping constant $c$ of a cylindrical dashpot with a cylindrical piston

The derivation for a cylindrical dashpot of constant  $c$  that is presented here in section A.1.0 is adopted from [5]. The derivation for the design presented in section 2.0 above that follows in section A.2.0 is derived in lines of that presented in [5].

The damping constant  $c$  of the dashpot can be determined using the shear stress equation for viscous fluid flow and the rate of fluid flow equation. As shown in the Figure 2 below, the dashpot contains a cylindrical piston of diameter  $D$  and length  $l$ , moving with velocity  $v_0$  inside a cylinder with a clearance  $t_0$ . The cylinder is filled with a fluid of viscosity  $\mu$ .

At a distance  $y$  from the moving surface, let the velocity and shear stress be  $v$  and  $\tau$ .

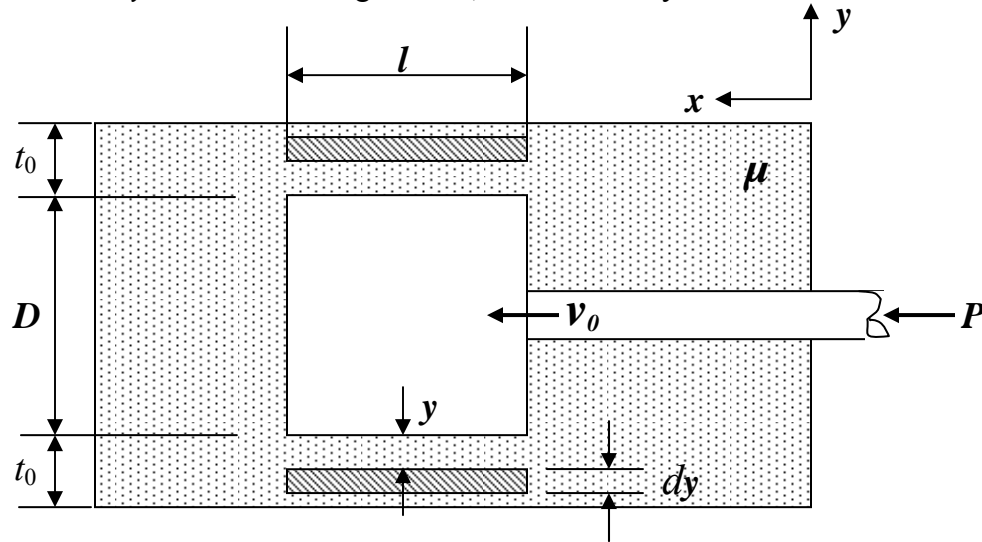


Figure 9 Cylindrical dashpot showing an annular ring around the cylindrical piston

Consider an element forming annulus of two cylinders of radius  $(D/2) + y$ , thickness  $dy$ . Let the velocity and shear stress at this distance be  $v - dv$  and  $\tau + d\tau$  respectively. The negative sign for  $dv$  shows that velocity decreases from the moving surface to the stationary cylinder. The elemental force  $dF$  on this annular ring is

$$dF = \pi D l d\tau = \pi D l \frac{d\tau}{dy} dy \quad \dots\dots\dots(1)$$

But, the shear stress  $\tau$  is given by  $\tau = -\mu \frac{dv}{dy}$

Therefore,  $dF = \pi D l \frac{d\tau}{dy} dy = -\pi D l \mu \frac{d^2v}{dy^2} dy \quad \dots\dots\dots(2)$

The force  $P$  on the piston causes a pressure difference  $p$  across the ends of the element, given by

$$p = \frac{P}{\pi D^2} = \frac{4P}{\pi D^2} \quad \dots\dots\dots (3)$$

Thus, the pressure force on the end of this element is given by  $p \cdot (\pi D dy) = \frac{4P}{D} dy$  .....(4)

where  $(\pi D dy)$  denotes the annular area between  $y$  and  $(y + dy)$  on the end face of the element.

Assuming uniform velocity in the direction of motion of the liquid, the forces given in (2), (3) must be equal.

Thus,

$$\frac{4P}{D} dy = -\pi D l \mu \frac{d^2 v}{dy^2} \cdot dy \quad \text{.....(5)}$$

$$\text{or} \quad \frac{d^2 v}{dy^2} = -\frac{4P}{\pi D^2 \mu l}$$

Integrating this equation twice under the limits  $v = v_0$  at  $y = y_0$  and  $v = 0$  at  $y = t_0$ ,

$$\text{we get} \quad v = \frac{2P}{\pi D^2 \mu l} (y t_0 - y^2) - v_0 \left( 1 - \left( \frac{y}{t_0} \right) \right) \quad \text{.....(6)}$$

The flow rate  $Q$  through the clearance space is obtained by integrating the rate of flow through an element between the limits  $y = 0$  and  $y = t_0$

$$Q = \int_0^{t_0} v \cdot (\pi D dy) = \pi D \left[ \frac{2P t_0^3}{6\pi D^2 l \mu} - \frac{1}{2} v_0 t_0 \right] \quad \text{.....(7)}$$

The volume of the liquid flowing through the clearance space must equal the volume displaced per second by the piston. But, this is equal to  $Q' = \frac{\pi D^2}{4} v_0$

$$\text{Therefore, setting } Q = Q', \quad \frac{\pi D^2}{4} v_0 = \pi D \left[ \frac{2P t_0^3}{6\pi D^2 l \mu} - \frac{1}{2} v_0 t_0 \right]$$

$$\Rightarrow P = \left[ \frac{3\pi \mu l D^3}{4 t_0^3} \cdot \left( 1 + \frac{2t_0}{D} \right) \right] v_0 \quad \text{But, } P = c v_0$$

$$\therefore c = \frac{3\pi \mu l D^3}{4 t_0^3} \cdot \left( 1 + \frac{2t_0}{D} \right) \quad \text{.....(8)}$$

Note:

1. The pressure keeps changing along the length of the piston. The pressure difference given in (3) is across the full length  $l$ .

**A.2.0 Derivation of the expression for damping constant  $c$  of a conical dashpot containing cylindrical piston**

The derivation of the expression for damping constant  $c$  of the design described in section 2.0 is done here in three sections A.2.1, A.2.2, A.2.3 as follows:

**A.2.1 Effective  $c$  when many pistons of damping constants  $c_i$  ( $i = 1,2,3,\dots$ ) are in series inside same cylinder of the dashpot**

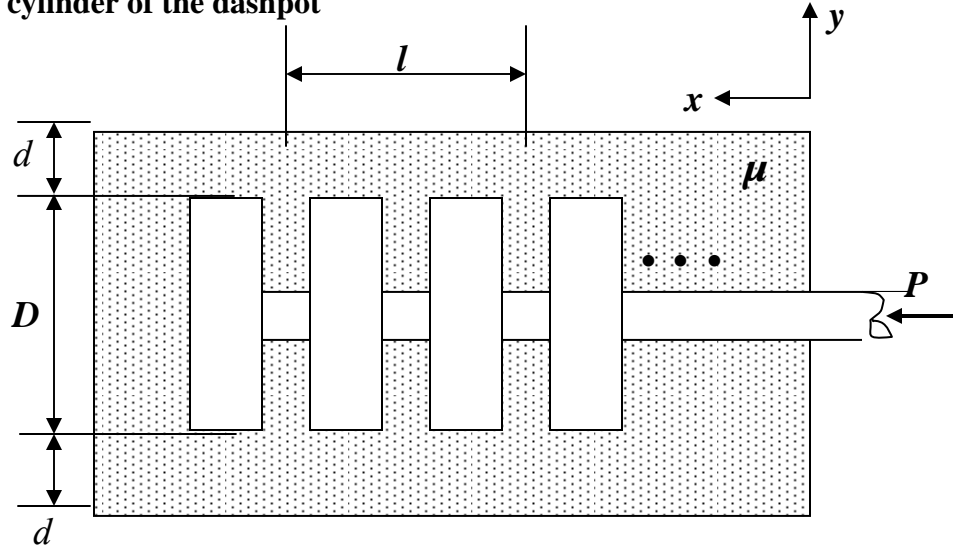


Figure 10 Multiple pistons in series inside same dashpot

In case of multiple pistons as shown in Figure 10 above, net friction force (Equation (1)) on the piston rod = sum of the frictional forces on each piston head. Hence, in effect, the combined  $c$  is equal to sum of all the individual  $c_i$  s. Therefore,  $c = c_1 + c_2 + c_3 \dots$  ..... (8)

**A.2.2 Considering the effect of imaginary fluid element at any section of cylindrical piston**

Now, at any section S of the cylindrical piston, consider an imaginary fluid element of vanishing length  $dz$  and of properties same as surrounding liquid.

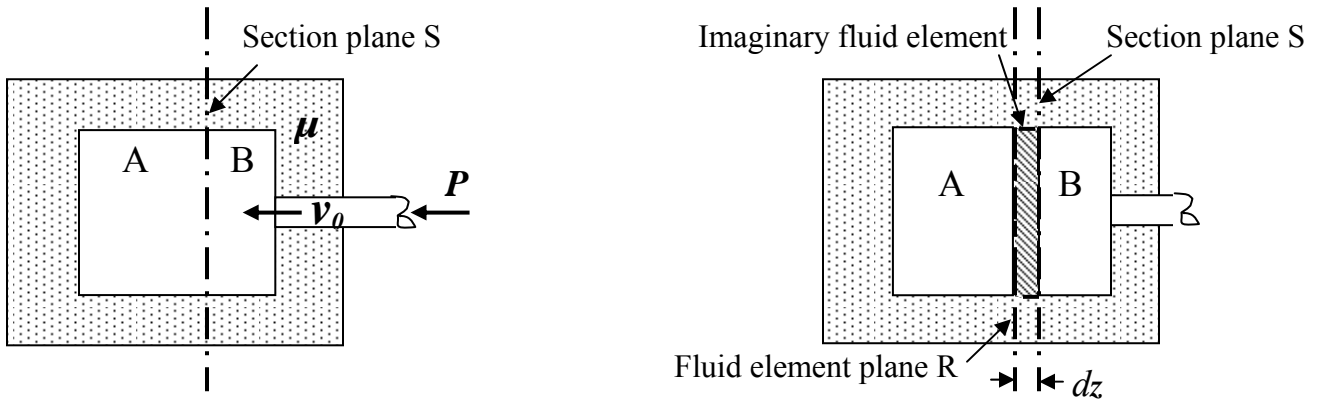


Figure 11 Introducing an imaginary fluid element  $dz$  of same properties as of surrounding liquid at the section plane S

When sectioned at S, each of the parts A and B of the piston experience a normal reaction  $N$ . This is shown below in Figure 12

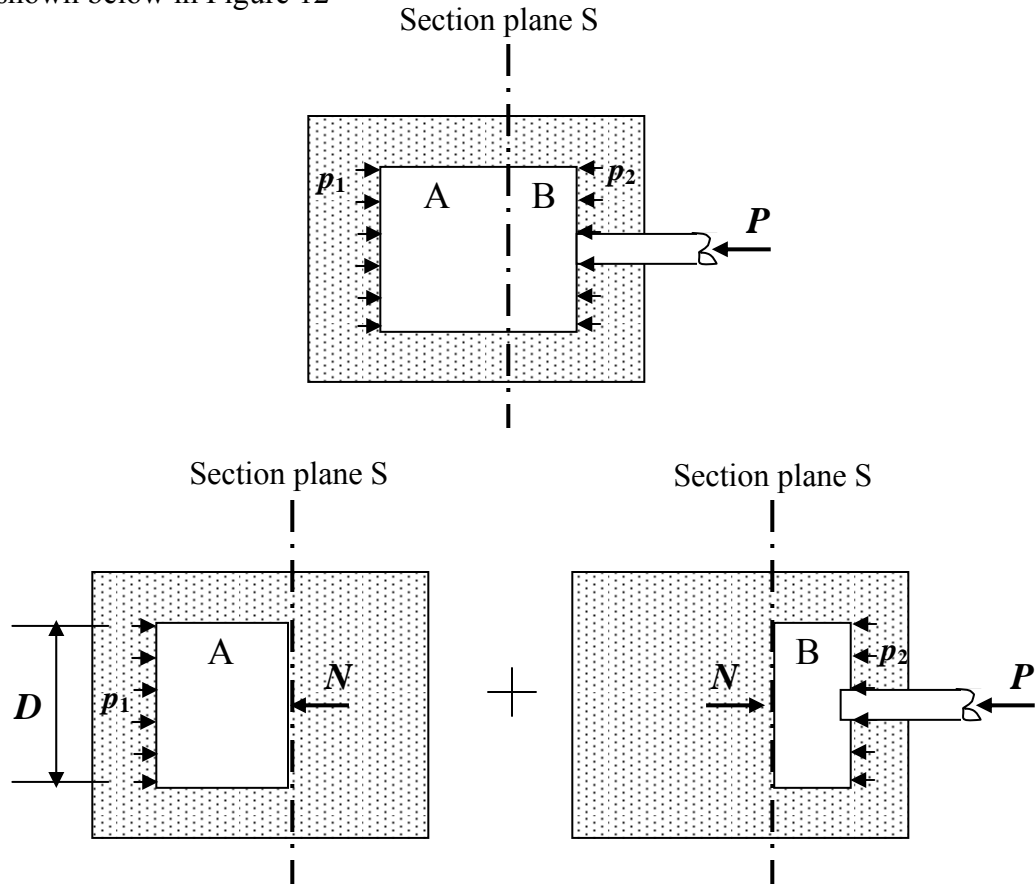


Figure 12 Piston sectioned at arbitrary plane S and normal reaction between the two parts shown

Now, consider the new fluid element of length  $dz$  at this plane S (Figure 13). Let the pressure in this fluid element be such that net force on each part A, B turns up as  $N$ . That is, the

pressure in this fluid element shall be  $p = \frac{N}{\frac{\pi D^2}{4}}$  so that net force =  $p \cdot \frac{\pi D^2}{4} = N$

In essence, the effect of introducing a fluid element with pressure this  $p$  is that the normal reaction  $N$  can now be removed, given that net force acting on the piston parts A, B are still the same and hence, the equations of motion do not change. Therefore, introduction of this fluid element does not change the physics of the problem – it is just the normal reaction between 2 parts of the piston looked upon as created by hydrostatic force from a liquid.

Therefore, a continuous piston can now be split into two parts A, B at any arbitrary section while introduced with a vanishing fluid element that has

pressure = (Normal reaction at this section) /  $\frac{\pi D^2}{4}$  so that the net effect is the same. .... (9)

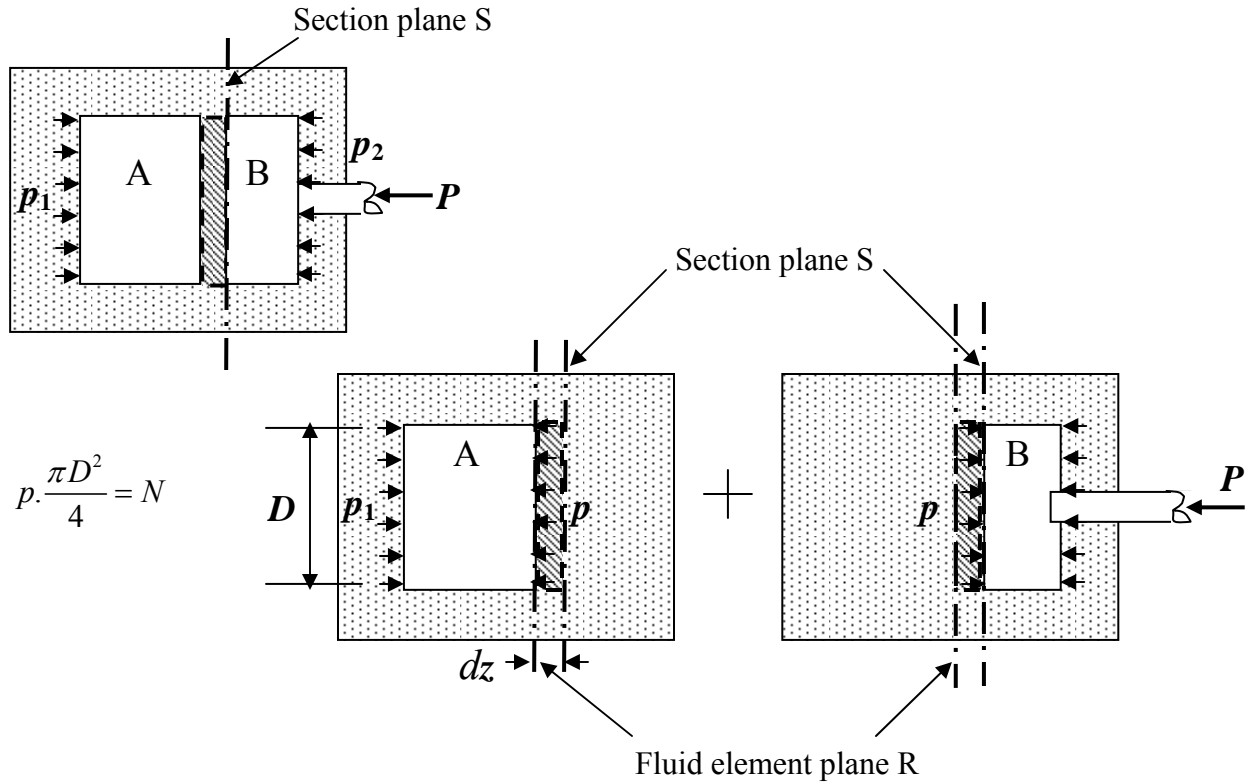


Figure 13 Imaginary fluid element (cross hatched) replacing the effect of normal reaction existing between the two parts of the piston A, B

### A.2.3 Expression for $c$ for conical dashpot with a cylindrical piston

Now, the derivation of the expression of  $c$  for conical dashpot with cylindrical piston is presented below: Figure 2 is revisited again. At a given displacement  $x$ , snapshot is presented here and the expression for  $c$  is derived.

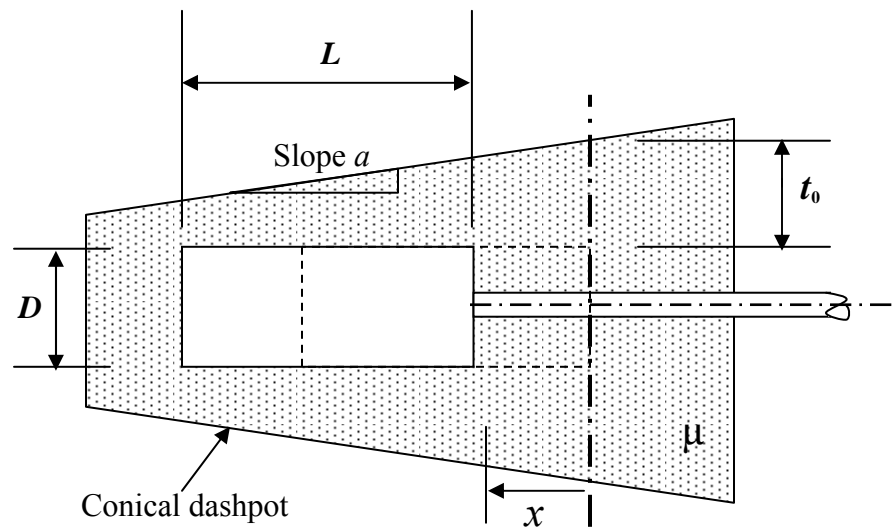


Figure 2 Conical dashpot with a cylindrical piston – schematic design

Now, conical dashpot is sectioned into infinitesimal elements, while placing at each section, the imaginary fluid element as described in A.2.2 as shown in Figure 14 below.

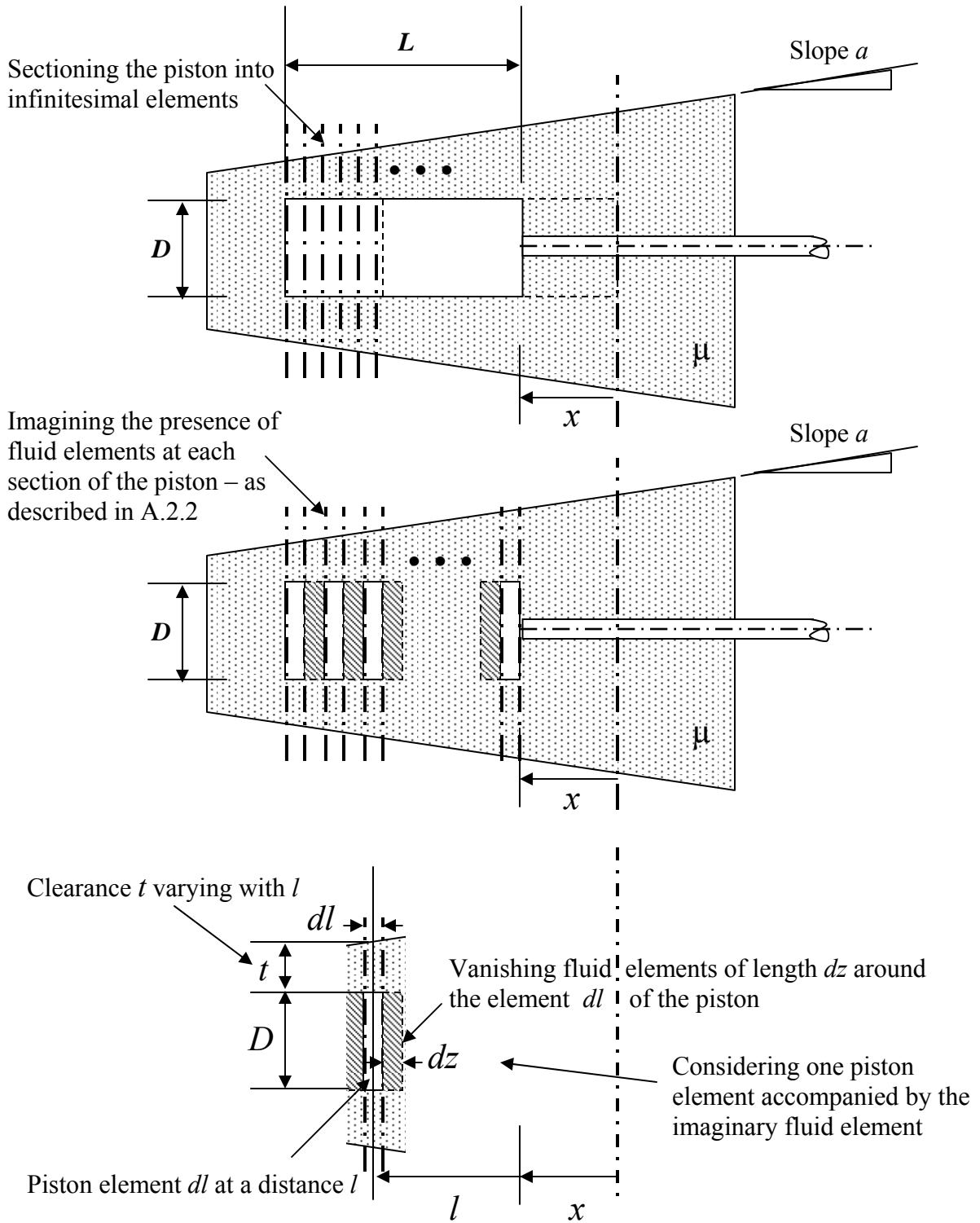


Figure 14 Conical dashpot with a cylindrical piston showing the piston elements obtained after sectioning and also, the accompanying imaginary fluid elements

This configuration shown in Figure 14 above can be imagined as many infinitesimal piston elements in series surrounded by vanishing fluid elements. As described in section A.2.2, presence of vanishing fluid elements does not change the physics of the problem (Equation (9)). All these piston elements are now in series, hence from equation (8) in A.2.1, the combined  $c$  is sum of all infinitesimal  $dc$  s.

The derivation of the damping constant  $dc$  of this infinitesimal piston element is all similar to that done in section A.1.0. Except that, in (1), instead of length  $l$ , since the piston is now an infinitesimal element of length  $dl$ , the elemental frictional force is

$$dF = \pi D . dl . d\tau = \pi D dl \frac{d\tau}{dy} . dy \quad \dots\dots\dots(10)$$

And in equation (3), the force to be considered is  $dP$  which gives a pressure difference  $dp$  given by

$$dp = \frac{dP}{\frac{\pi D^2}{4}} = \frac{4dP}{\pi D^2} \quad \dots\dots\dots (11)$$

All these elemental forces  $dP$  add-up to the force  $P$  applied at the end of the piston.

That is, in the fluid element to the right of the piston element (Figure 15), if the pressure is  $p$ , then, the pressure to be imagined in the fluid element to the left of piston element is  $(p - dp)$  where  $dp$  is given by (11).

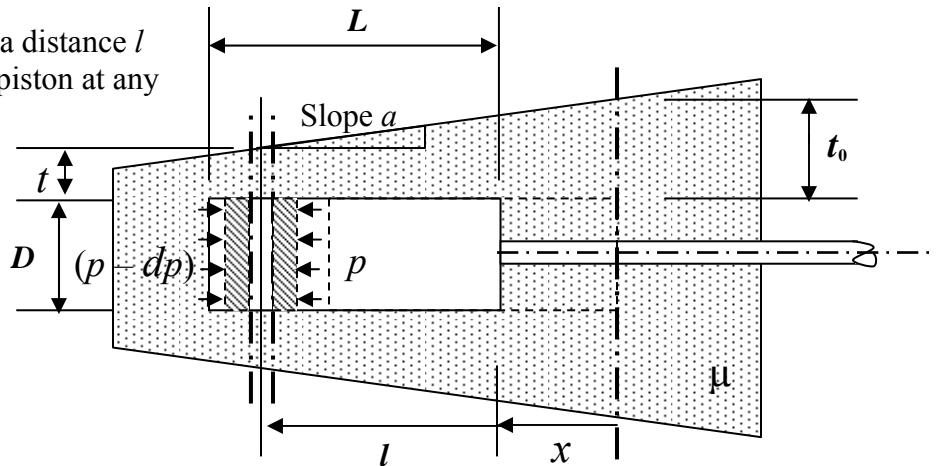
All the further steps viz. equations (4) through (6) are repeated, while in (7), the integration is done from limits 0 to  $t$ , where  $t$  is the clearance at this element. Hence, going through the steps after (7), the expression for infinitesimal damping constant  $dc$  for the piston element is given by

$$dc = \frac{3\pi\mu D^3}{4t^3} \cdot \left(1 + \frac{2t}{D}\right) . dl \quad \dots\dots\dots(12)$$

From (8), all these  $dc$  s can be added, in other words, integrated to get expression for full  $c$ .

But, here,  $t$  varies with  $l$  due to dashpot's surface being conical. If the slope of the conical surface is  $a$ , then, at any given stroke  $x$ ,  $t(l) = t_0 - a . (l + x)$  ..... (13)

Figure 15 Clearance  $t$  at a distance  $l$  from the rear end of the piston at any given displacement  $x$



Note that the length  $dz$  of vanishing fluid element is not considered in (13) since even if it were, when limit  $dz \rightarrow 0$  is applied, it will disappear.

$$\therefore c = \int_0^L dc = \int_0^L \frac{3\pi\mu D^3}{4t^3} \cdot \left(1 + \frac{2t}{D}\right) \cdot dl$$

From (13),  $dt = -a \cdot dl \Rightarrow dl = \frac{-dt}{a}$  and limits are  $l = 0, t = t_0 - ax$ ;  $l = L, t = t_0 - a(L+x)$ ;

$$\Rightarrow c = \int_{t_0-ax}^{t_0-a(L+x)} \frac{3\pi\mu D^3}{4t^3} \cdot \left(1 + \frac{2t}{D}\right) \cdot \frac{-dt}{a}$$

$$\therefore c = \frac{3\pi\mu D^3}{4a} \left[ \frac{1}{2} \left( \frac{1}{(t_0 - a(L+x))^2} - \frac{1}{(t_0 - ax)^2} \right) + \frac{2}{D} \left( \frac{1}{t_0 - a(L+x)} - \frac{1}{t_0 - ax} \right) \right] \dots\dots\dots (14)$$

Check:

When the slope  $a$  of the conical surface tends to zero, that is, when the dashpot becomes cylindrical, (14) must give (8).

$$\therefore \lim_{a \rightarrow 0} c = \lim_{a \rightarrow 0} \frac{3\pi\mu D^3}{4a} \left[ \frac{1}{2} \left( \frac{1}{(t_0 - a(L+x))^2} - \frac{1}{(t_0 - ax)^2} \right) + \frac{2}{D} \left( \frac{1}{t_0 - a(L+x)} - \frac{1}{t_0 - ax} \right) \right]$$

This is of  $\frac{0}{0}$  form. Hence, applying *L'Hospital's* rule,

$$\begin{aligned} \lim_{a \rightarrow 0} c &= \frac{3\pi\mu D^3}{4} \lim_{a \rightarrow 0} \frac{1}{\frac{d}{da}(a)} \cdot \frac{d}{da} \left[ \frac{1}{2} \left( \frac{1}{(t_0 - a(L+x))^2} - \frac{1}{(t_0 - ax)^2} \right) + \frac{2}{D} \left( \frac{1}{t_0 - a(L+x)} - \frac{1}{t_0 - ax} \right) \right] \\ &= \frac{3\pi\mu D^3}{4} \lim_{a \rightarrow 0} \left[ \left( \frac{(L+x)}{(t_0 - a(L+x))^3} - \frac{x}{(t_0 - ax)^3} \right) + \frac{2}{D} \left( \frac{(L+x)}{(t_0 - a(L+x))^2} - \frac{x}{(t_0 - ax)^2} \right) \right] \\ &= \frac{3\pi\mu D^3}{4} \cdot \left[ \left( \frac{(L+x)}{(t_0 - 0 \cdot (L+x))^3} - \frac{x}{(t_0 - 0 \cdot x)^3} \right) + \frac{2}{D} \left( \frac{(L+x)}{(t_0 - 0 \cdot (L+x))^2} - \frac{x}{(t_0 - 0 \cdot x)^2} \right) \right] \\ &= \frac{3\pi\mu D^3}{4} \cdot \left[ \left( \frac{(L+x)}{(t_0)^3} - \frac{x}{(t_0)^3} \right) + \frac{2}{D} \left( \frac{(L+x)}{(t_0)^2} - \frac{x}{(t_0)^2} \right) \right] \\ &= \frac{3\pi\mu D^3}{4} \cdot \left[ \left( \frac{L+x-x}{(t_0)^3} \right) + \frac{2}{D} \left( \frac{L+x-x}{(t_0)^2} \right) \right] = \frac{3\pi\mu D^3 L}{4t_0^3} \left[ 1 + \frac{2t_0}{D} \right] \\ \therefore c' = \lim_{a \rightarrow 0} c &= \frac{3\pi\mu D^3 L}{4t_0^3} \left[ 1 + \frac{2t_0}{D} \right] \text{ which is same as (8)} \end{aligned}$$

This completes the derivation of expression for  $c$  for conical dashpot.